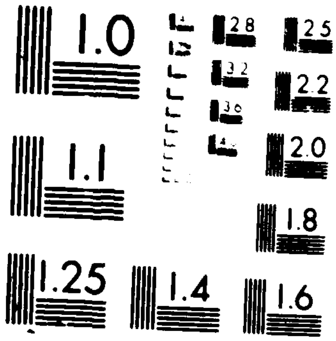
AD-A186 014 ON THE FEYNMAN-KAC'S FORMULA AND ITS APPLICATIONS TO FILTERING THEORY (U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROC R L KARANDIKAR 1/1

UNCLASSIFIED OCT 86 TR-161 AFOSR-TR-87-1096 F/G 12/3 NL



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AD-A186 014

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED		1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY: NA		3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for Public Release; Distribution Unlimited	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: NA			
4. PERFORMING ORGANIZATION REPORT NUMBER(S)		5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 87-1096	
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina	6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM	
6c. ADDRESS (City, State and ZIP Code): Center for Stochastic Processes, Statistics Department, Phillips Hall 039-A, Chapel Hill, NC 27514		7b. ADDRESS (City, State and ZIP Code): Bldg. 410 Bolling AFB, DC 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR	8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: F49620 85 C 0144	
8c. ADDRESS (City, State and ZIP Code): Bldg. 410 Bolling AFB, DC		10. SOURCE OF FUNDING NOS.	

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
6.1102F	2304	A5	

11. TITLE (Include Security Classification): ON THE FEYNMAN-KAC'S FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

12. PERSONAL AUTHOR(S): R.L. Karandikar

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
~~Technical~~ Journal	9/30/85-9/30/86	Oct. 86	21

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB. GR.	Keywords:
XXXXXXX	XXXXXXX	XXXXX	

19. ABSTRACT

Let (X_t) be a Markov process, not assumed to be time homogeneous. It is well known that $\hat{X}_t = (t, X_t)$ is a time homogeneous Markov process. Let A be its generator. The Feynman-Kac's formula for X_t takes the following form if the equation

(1.1) $$Av + cv = 0$$

admits a solution v, then v has the representation, for $s < t$

(1.2) $$v(s, X_s) = E\left[v(t,X_t)\exp\left(\int_s^t c(u,X_u)du\right)\Big|\sigma(X_s)\right].$$

We prove this under general conditions on (X_t).

DTIC SELECTED OCT 07 1987 E

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED

22a. NAME OF RESPONSIBLE INDIVIDUAL	22b. TELEPHONE NUMBER (Include Area Code)	22c. OFFICE SYMBOL
~~Peggy Ravitch~~ Major Crowley	~~919-962-2307~~	AFOSR/NM

CENTER FOR STOCHASTIC PROCESSES

AFOSR-TR- 87-1096

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Technical Report No. 161

October 1986

ON THE FEYNMAN-KAC's. FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

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Technical Report No. 161 (UNC)

Technical Report No. 8408 (ISI)

Research partially supported by the Air Force Office of Scientific Research Contract #F49620 85 C 0144 and by the Indian Statistical Institute.

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1. Introduction : Let (X_t) be a Markov process, not assumed to be time homogeneous. It is well known that $\hat{X}_t = (t, X_t)$ is a time homogeneous Markov process. Let A be its generator. The Feynman-Kac's formula for X_t takes the following form if the equation

$$(1.1) \qquad Av + cv = 0$$

admits a solution v, then v has the representation, for $s < t$

$$(1.2) \qquad v(s, X_s) = E\left[v(t,X_t)\exp\left(\int_s^t c(u,X_u)du\right)\middle|\sigma(X_s)\right].$$

We prove this under general conditions on (X_t).

Then we come to the question of existence of solution to (1.1). We show that under some regularity conditions on (X_t), (1.1) has a solution for a rich class of boundary conditions. This implies that the 'dual' equation to (1.1) admits a unique solution. The 'dual' equation is an equation for measures on the state space of (X_t) and its unique solution is the distribution of X_t under an absolutely continuous change of the underlying probability measure by a multiplicative functional.

These results on the measure valued equations significantly extend results given in [3] on the conditional distributions for the nonlinear filtering problem (in the white noise approach).

2. Let $(S, \underline{\underline{S}})$ be a measurable space. Let (X_t) be an $(S, \underline{\underline{S}})$ valued Markov process on a probability space $(\Omega, \underline{\underline{A}}, \pi)$ with transition probability function P, i.e.

$$\{\omega : X_t(\omega) \in B\} \in \underline{\underline{A}}$$

and

(2.1) $$E_\pi\left[1_B(X_t) \mid \underline{\underline{F}}_s^X\right] = P(s, X_s, t-s, B) \quad \text{a.s. } \pi$$

for all $0 \leq s \leq t < \infty$, $B \in \underline{\underline{S}}$. Here, the function $P(s,x,t,B)$ on $\{0 \leq s < \infty, t \geq 0, x \in S, B \in \underline{\underline{S}}\}$ is assumed to satisfy the following conditions.

(2.2) For $s \geq 0$, $t \geq 0$, $x \in S$; $P(s,x,t,.)$ is a countably additive probability measure on $(S, \underline{\underline{S}})$.

(2.3) For $s \geq 0$, $x \in S$, $B \in \underline{\underline{S}}$; $P(s,x,0,B) = 1_B(x)$.

(2.4) For $t \geq 0$, $B \in \underline{\underline{S}}$; $(s,x) \rightarrow P(s,x,t,B)$ is a $B([0,\infty)) \otimes \underline{\underline{S}}$ measurable function ($\underline{\underline{B}}(E)$ denotes the Borel σ-field of a topological space E and $\otimes$ denotes the product of σ-fields).

(2.5) For $s \geq 0$, $u \geq 0$, $t \geq 0$, $x \in S$, $B \in \underline{\underline{S}}$; we have

$$\int_S P(s+t,z,u,B)\, P(s,x,t,dz) = P(s,x,t+u,B)$$

Throughout, $\underline{\underline{F}}_t^X$ denotes the smallest σ-field with respect to which the family $\{X_u : 0 \leq u \leq t\}$ is measurable. We also assume that

(2.6) the process (X_t) is $\underline{\underline{F}}_t^X$ - progressively measurable, i.e. for all $t_0 < \infty$, the mapping $(t,w) \to X_t(\omega)$ from $[0,t_0] \times \Omega \to S$ is $\underline{\underline{B}}([0,t_0]) \otimes \underline{\underline{F}}_{t_0}^X$ measurable.

Let $\hat{S} = [0,\infty) \times S$, $\hat{\underline{\underline{S}}} = \underline{\underline{B}}([0,\infty)) \otimes \underline{\underline{S}}$ and $\underline{\underline{J}}$ be the class of bounded real valued $\hat{\underline{\underline{S}}}$ measurable functions on $\hat{S}$.

<u>Definition</u> : A sequence $\{f_k\} \subseteq \underline{\underline{J}}$ is said to converge weakly to $f \in \underline{\underline{J}}$, written as $\underset{k \to \infty}{\text{w-lim}}\ f_k = f$, if $f_k(x)$ is uniformly bounded and for each $x \in S$, $f_k(x)$ converges to $f(x)$.

For $f \in \underline{\underline{J}}$, $t \geq 0$, let $T_t f : S \to \mathbb{R}$ be defined by

$$(2.7) \qquad (T_t f)(s,x) = \int f(s+t,z)\, P(s,x,t,dz), \quad (s,x) \in \hat{S}.$$

Using the properties of P, it can be checked that $T_t f \in \underline{\underline{J}}$ and that for $u \geq 0$, $t \geq 0$,

$$(2.8) \qquad T_u \left[T_t f\right] = T_{t+u} f \quad , \qquad f \in \underline{\underline{J}} \ .$$

Thus $\{T_t : t \geq 0\}$ is a semigroup of operators (from $\underline{\underline{J}}$ into itself).

<u>Remark</u> : It is well known and easy to check that $\hat{X}_t = (t, X_t)$ is a Markov process with stationary transition probability function $\hat{P}$ given by

$$\hat{P}(t, (s,x), B) = P(s,x,t,B^{(s+t)}), \quad B \in \hat{\underline{\underline{S}}}$$

where B^u denotes the u-section of $B \subseteq S$. The semigroup $\{T_t : t \geq 0\}$ defined above is the usual semigroup associated with the transition function P (as in [2], section 2.1).

We will now recall the definition and some properties of the weak generator A of $\{T_t : t \geq 0\}$. Let $\underline{\underline{J}}_o$ be given by

$$\underline{\underline{J}}_o = \{f \in \underline{\underline{J}} \; ; \; \underset{t \downarrow 0}{\text{w-lim}} \; T_t f = f\}$$

<u>Definition</u> : Let $\underline{\underline{D}}_A$ be the class of $f \in \underline{\underline{J}}$ for which the

$$\text{(2.9)} \qquad \underset{t \downarrow 0}{\text{w-lim}} \; \frac{T_t f - f}{t} = g$$

exists and belongs to $\underline{\underline{J}}_o$ and for $f \in \underline{\underline{D}}_A$, define $Af = g$, where g is given by (2.9).

The following properties are easy to prove. We will only state them here. For a proof see chapter 1 in [2].

(2.10) $\quad T_t(\underline{\underline{D}}_A) \subseteq \underline{\underline{D}}_A$ and for $f \in \underline{\underline{D}}_A$, $A(T_t f) = T_t Af$

(2.11) For $f \in \underline{\underline{J}}_o$, $t \to (T_t f)(s,x)$ is a right continuous function for all $(s,x) \in \hat{S}$.

(2.12) For $f \in \underline{\underline{D}}_A$, we have, for all $(s,x) \in \hat{S}$, $t \geq 0$

$$(T_t f)(s,x) = f(s,x) + \int_0^t (T_u Af)(s,x)du$$

(2.13) Given $f \in \underline{\underline{J}}_o$ there exists a sequence $\{f_k\} \subseteq \underline{\underline{D}}_A$ such that

$$\underset{k \to \infty}{\text{w-lim}} \; f_k = f.$$

In (2.13) above, f_k can be taken to be

$$f_k(s,x) = \int_0^\infty k\, e^{-kt}\, (T_t\, f)(s,x)dx \;.$$

The property (2.12) has the following important consequence.

<u>Proposition 1</u> : For $f \in \underline{\underline{D}}_A$, M_t given by

$$M_t(\omega) = f(t, X_t(\omega)) - \int_0^t (Af)(u, X_u(\omega))du \tag{2.14}$$

is a martingale with respect to the σ-fields $\underline{\underline{F}}_t^X$.

<u>Proof</u> : The progressive measurability of (X_t) implies the $\underline{\underline{F}}_t^X$ measurability of M_t. Since f, $Af \in \underline{\underline{J}}$, they are bounded and hence M_t is itself bounded for each t. Now (2.1) implies

$$\begin{aligned} E_\pi\left[f(t, X_t) | \underline{\underline{F}}_s^X\right] &= \int f(t,z)\, P(s, X_s, t-s, dz) \\ &= (T_{t-s}\, f)\, (s, X_s) \end{aligned} \tag{2.15}$$

for $s \leq t$. Similarly for $s \leq u$, we have

$$E_\pi\left[(Af)(u,X_u) | \underline{\underline{F}}_s^X\right] = (T_{u-s}\, Af)(s, X_s). \tag{2.16}$$

Using (2.11), (2.12), (2.15) and (2.16), it can be checked that

$$E_\pi\left[M_t - M_s | \underline{\underline{F}}_s^X\right] = 0 \;.$$

$\square$

We now turn our attention to the Feynman-Kac's formula. Our next result is a step in this direction.

Let $g : [0,t_o] \times S \to \mathbb{R}$ be a $\underline{\underline{B}}([0,t_o]) \times \underline{\underline{S}}$ measurable function such that

$$(2.17) \qquad E_\pi\left[\int_0^{t_o} |g(u, X_u)|\,du\right] < \infty$$

and for a positive integrable function $a : [0, t_o] \to \mathbb{R}$,

$$(2.18) \qquad g(u, x) \leq a(u) \quad \text{for all } x \in S, \; u \in [0, t_o].$$

Fix $0 \leq s \leq t_o$ and let

$$(2.19) \qquad B_t(\omega) = \exp\left(\int_s^t g(u,X_u(\omega))du\right).$$

Then we have

<u>Theorem 2</u> : Let $f \in \underline{\underline{D}}_A$ and g satisfy (2.17), (2.18). Then

$$(2.20) \qquad Z_t = f(t, X_t)\cdot B_t - \int_s^t \left[(Af)(u,X_u) + g(u,X_u)\right]\cdot B_u\,du$$

is an $\underline{\underline{F}}_t^X$ martingale for $t \geq s$ (where B is given by (2.19)).

<u>Proof</u> : It is easy to see that Z_t is $\underline{\underline{F}}_t^X$ measurable. The condition (2.18) implies that B_t is bounded. Since f, Af are also bounded the condition (2.17) gives the integrability of Z_t. To prove the martingale property, suffices to prove that for $s \leq r \leq t$, $C \in \underline{\underline{F}}_r^X$,

$$(2.20) \qquad E_\pi\left[(Z_t - Z_r)\cdot 1_C\right] = 0 \; .$$

Let $f_1(t,\omega) = f(t_o, X_{t_o}(\omega)) - \int_t^{t_o} (Af)(u,X_u(\omega))du$. Then by Proposition 1, it follows that for $0 \leq t \leq t_o$

$$(2.21) \qquad E_\pi \left[f_1(t,.) \mid \underline{F}_t^X \right] = f(t,X_t)$$

and hence

$$(2.22) \qquad E_\pi \left[1_C \cdot (Z_t - Z_r) \right] = E_\pi \, 1_C \cdot \{ f_1(t,.) B_t - f_1(r,.) B_r - \int_r^t ((Af+gf)(u,X_u) du \}.$$

Now for each ω, $f_1(t,\omega)$, $B_t(\omega)$ are absolutely continuous functions and hence

$$\begin{aligned} f_1(t,\omega)B_t(\omega) - f_1(r,\omega)B_r(\omega) &= \int_r^t \frac{d}{du}\left[f_1(u,\omega)B_u(\omega) \right] du \\ &= \int_r^t \{ f_1(u,\omega) \cdot g(u,X_u(\omega)) B_u(\omega) \\ &\quad + (Af)(u,X_u(\omega)) . B_u(\omega) \} du. \end{aligned}$$

Thus

$$\begin{aligned} (2.23) \qquad E_\pi \left[f_1(t,.)B_t - f_1(r,.)B_r \right] &= E \, 1_C \int_r^t f_1(u,.) g(u,X_u) B_u \, du \\ &\quad + E \, 1_C \int_r^t (Af)(u,.) B_u \, du \\ &= E \, 1_C \int_r^t (Af+gf)(u,X_u) du \end{aligned}$$

using (2.21) once again. Now (2.22) and (2.23) give the required euqlity

$$E \left[1_C (Z_t - Z_r) \right] = 0 .$$

Remark : It can be verified that

$$Z_t = M_t \, B_t - \int_s^t M_u \, dB_u$$

where M is given by (2.14). Hence if M were right continuous, it would follow from the "integration by parts formula for martingale"

(See [5]) that $(Z_t, \underline{\underline{F}}_t^X)$ is a martingale. However, in general M_t need not be right continuous and hence we have given a direct proof.

The following is the Feynman-Kac's formula for a time inhomogeneous Markov process.

<u>Theorem 3</u> : Let $0 < t_0 < \infty$ be fixed. Let $c : [0, t_0] \times S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be bounded measurable functions. Suppose that $v \in \underline{\underline{D}}_A$ is a solution to

$$(2.24) \qquad [\Lambda v + cv](u,x)\, 1_{\{u < t_0\}} = 0$$

and

$$(2.25) \qquad v(t_0,x) = g_0(x) .$$

Then v admits a representation, for $s < t_0$

$$(2.26) \qquad v(s,X_s) = E_\pi \left[g_0(X_{t_0}) \exp\left(\int_s^{t_0} c(u,X_u)du\right) \middle| \underline{\underline{F}}_s^X \right] \quad \text{a.e. } \pi.$$

<u>Proof</u> : Fix $s < t_0$. Take $f = v$ and $g = c$ in Theorem 2 to obtain that $(Z_t, \underline{\underline{F}}_t^X)_{s \leq t \leq t_0}$ is a martingale, where

$$(2.27) \qquad Z_t = v(t,X_t) \exp\left(\int_s^t v(u,X_u)du\right).$$

Here we have used the fact that v satisfies (2.24) so that the second term appearing in the expression for Z_t is zero. Thus

$$E_\pi [Z_{t_0} | \underline{\underline{F}}_s^X] = Z_s \qquad \text{a.s. } \pi$$

This is same as (2.26) since $v(t_0,x) = g_0(x)$.

$\square$

3. In this section we consider the question as to under what conditions on $c, g, (X_t)$ does the problem (2.24), (2.25) admit a solution. Of course, if the solution exists, it has to satisfy (2.26) and this gives a clue as to what conditions one should put on $c, g, (X_t)$.

Suppose that S is a topological space, $\underline{\underline{S}}$ is its Borel σ field. Let $\mathcal{X}$ be the space of all right continuous mappings $\underline{X}$ from $[0,\infty)$ into S. We will denote by $\underline{X}_t$ the value of $\underline{X}$ at t. Let $\underline{\underline{A}}_t^s = \sigma(\underline{X}_u : s \le u \le t)$. We assume that

$$(3.1) \qquad \text{for all } \omega, \; X_\cdot(\omega) \in \mathcal{X}$$

and that for all $(s,x) \in \hat{S}$, there exists a probability measure $P_{s,x}$ on $(\mathcal{X}, \underline{\underline{A}}_\infty^s)$ such that for $0 \le t_0 \le t_1 \le \dots \le t_k, y_0 \in S$, $A_1, A_2, \dots, A_k \in \underline{\underline{S}}$; $k \ge 1$, we have

$$(3.2) \qquad P_{t_0,y_0}(\underline{X}_{t_i} \in A_i : 1 \le i \le k) = \int \dots \int \prod_{i=1}^{k} 1_{A_i}(y_i) P(t_{i-1}, y_{i-1}, t_i - t_{i-1}, dy_i).$$

<u>Remark</u> The main thrust of this assumption is that $P_{s,x}$ is realized on $\mathcal{X}$. The relation (2.1) and (3.2) imply that for $\{t_i\}, \{A_i\}$ as in (3.2), we have

$$(3.3) \qquad E_\pi\left[\prod_{i=1}^{k} 1_{A_i}(X_{t_i}) \,\middle|\, \underline{\underline{F}}_{t_0}^X\right] = P_{t_0,\underline{X}_{t_0}}(\underline{X}_{t_i} \in A_i : 1 \le i \le k) \text{ a.s. } \pi$$

and hence by standard arguments, we have for $B \in \underline{\underline{A}}_\infty^{t_0}$,

$$(3.4) \qquad \pi(X_\cdot \in B \mid \underline{\underline{F}}_{t_0}^X) = P_{t_0, X_{t_0}}(B) \qquad \text{a.s. } \pi \; .$$

Similarly, it can be proved that for $s < t$, $B \in \underline{\underline{A}}_\infty^t$, $x \in S$,

$$(3.5) \qquad P_{s,x}(B \mid \underline{\underline{A}}_t^s) = P_{t,\underline{X}_t}(B) \qquad \text{a.s. } P_{s,x} \; .$$

We are now in a position to prove a 'converse' to the Feynman-Kac's formula.

Theorem 4 : Let $0 < t_o < \infty$ be fixed. Let $c : [0,t_o] \times S \to \mathbb{R}$ be a bounded continuous function. Let $f \in \underline{\underline{D}}_A$. Let $v : \hat{S} \to \mathbb{R}$ be defined by

$$(3.6) \qquad v(s,x) = E_{P_{s,x}}\left[f(t_o,X_{t_o})\exp(\int_s^{t_o} c(u,\underline{X}_u)du)\right] , \quad s < t_o$$

$$= f(s,x) \qquad , \quad s \geq t_o.$$

Then $v \in \underline{\underline{D}}_A$ and $Av = f_1$ where

$$(3.7) \qquad f_1(s,x) = -c(s,x)v(s,x) \quad , \quad s < t_o$$

$$= (Af)(s,x) \qquad s \geq t_o .$$

Proof Since $v(s,x) = f(s,x)$ for $s \geq t_o$, we have

$$(T_t v)(s,x) = (T_t f)(s,x)$$

for $s \geq t_o$, $x \in S$, $t \geq 0$. Hence for $s \geq t_o$, $x \in S$.

$$(3.8) \qquad \underset{t \downarrow 0}{\text{w-lim}} \frac{(T_t v)(s,x) - v(s,x)}{t} = (Af)(s,x) = f_1(s,x) .$$

For $\underline{X} \in \mathcal{X}$, $s \leq t_o$ let us define

$$(3.9) \qquad C_s(\underline{X}) = \exp(\int_s^{t_o} c(u, \underline{X}_u)du).$$

Then for $s \leq t_o$, we have

$$v(s,x) = E_{P_{s,x}}\left[f(t_o,X_{t_o}) C_s(\underline{X})\right] .$$

For $s < t_o$, $s + t < t_o$, we thus have

$$(3.10) \qquad (T_t\, v)(s,x) = \int v(s+t,z)\, P(s,x,t,dz)$$

$$= E_{P_{s,x}}\left[v(s+t, \underline{X}_{s+t})\right]$$

$$= E_{P_{s,x}}\left[E_{P_{s+t,\, \underline{X}_{s+t}}}(f(t_o,\underline{X}_{t_o})C_{s+t}(\underline{X}))\right]$$

$$= E_{P_{s,x}}\left[f(t_o, \underline{X}_{t_o})\, C_{s+t}(\underline{X})\right].$$

by (3.5). Hence, for $s < t_o$, $x \in S$, $s + t < t_o$ we have

$$(3.11) \qquad \frac{(T_t v)(s,x) - v(s,x)}{t} = E_{P_{s,x}}\left[f(t_o,\underline{X}_{t_o}) \cdot \frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t}\right].$$

For all $\underline{X} \in \mathfrak{X}$, we have from (3.9)

$$(3.12) \qquad \lim_{t \downarrow 0} \frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t} = -\, c(s,\underline{X}_s) \cdot C_s(\underline{X}).$$

Further

$$(3.13) \qquad \left|\frac{C_{s+t}(\underline{X}) - C_s(\underline{X})}{t}\right| = \left|-\, c(\tau,\underline{X}_\tau) \cdot C_\tau(\underline{X})\right|$$

$$\leq K$$

where K depends only on t_o and the upper bound of $|c|$. The dominated convergence theorem gives that for $s < t_o$

$$(3.14) \qquad \lim_{t \downarrow 0} \frac{(T_t v)(s,x) - v(s,x)}{t} = E_{P_{s,x}}\left[f(t_o,\underline{X}_{t_o})\{-c(s,\underline{X}_s)C_s(\underline{X})\}\right]$$

$$= -\, c(s,x)\, v(s,x)$$

$$= f_1(s,x)$$

as $P_{s,x}(\underset{\sim}{X}_s = x) = 1$. Also, (3.13) implies that the left hand expression in (3.11) is uniformly bounded (in s,x,t). Thus we have

$$\text{(3.15)} \qquad \underset{t \downarrow 0}{\text{w-lim}} \; \frac{(T_t v)(s,x) - v(s,x)}{t} = f_1(s,x) .$$

Remains to prove that $f_1 \in \underline{\underline{J}}_o$. This will prove that $v \in \underline{\underline{D}}_A$ and that $Av = f_1$. If $s \geq t_o$, $f_1(s,x) = (Af)(s,x)$ and hence for $s \geq t_o$, $t \geq 0$, $(T_t f_1)(s,x) = (T_t Af)(s,x)$. Since $Af \in \underline{\underline{J}}_o$, this gives

$$\text{(3.16)} \qquad \lim_{t \downarrow 0} (T_t f_1)(s,x) = f_1(s,x) \quad \text{for} \quad s \geq t_o, \quad x \in S .$$

For $s < t_o$, we have

$$T_t f_1(s,x) - f_1(s,x) = -E_{P_{s,x}}\left[c(s+t,\underset{\sim}{X}_{s+t})v(s+t,\underset{\sim}{X}_{s+t}) - c(s,x)v(s,x)\right]$$

$$= - E_{P_{s,x}}\left[v(s+t,\underset{\sim}{X}_{s+t}) \{c(s+t,\underset{\sim}{X}_{s+t}) - c(s,x)\}\right]$$

$$- c(s,x)\, E_{P_{s,x}}\left[v(s+t,\underset{\sim}{X}_{s+t}) - v(s,x)\right] .$$

Now as $t \downarrow 0$, $c(s+t, \underset{\sim}{X}_{s+t}) \to c(s,\underset{\sim}{X}_s) = c(s,x)$ a.e. $P_{s,x}$, as c is continuous and $\underset{\sim}{X}_u$ is right continuous. Hence by the dominated convergence theorem,

$$\lim_{t \downarrow 0} E_{P_{s,x}}\left[v(s+t, \underset{\sim}{X}_{s+t}) \{c(s+t, \underset{\sim}{X}_{s+t}) - c(s,x)\}\right] = 0.$$

The relation (3.14) implies that

$$-c(s,x)\, E_{P_{s,x}}\left[v(s+t, \underset{\sim}{X}_{s+t}) - v(s,x)\right] = - c(s,x)\left[(T_t v)(s,x) - v(s,x)\right]$$

$$\to 0$$

as $t \downarrow 0$. These observations give

$$(3.17) \qquad \lim_{t \downarrow 0} (T_t f_1)(s,x) = f_1(s,x) \quad \text{for} \quad s < t_o, x \in S \ .$$

Now (3.16), (3.17) and the fact that $T_t f_1$ is uniformly bounded yield

$$\text{w-lim}_{t \downarrow 0} (T_t f_1) = f_1 \quad .$$

□

Remark : Under the conditions assumed in this section and Theorem 4, the equations (2.24), (2.25) for $g_o(x) = f(t_o,x)$ have a unique solution v on $[0,t_o] \times S$ which is given by (3.6). To see this, let v' be any solution. Apply Theorem 3 to the process $\{\underline{X}_t : t \geq s\}$ on the probability space $(\mathfrak{X}, \underline{\underline{A}}^s_\infty, P_{s,x})$ to obtain, for $s < t_o$,

$$v'(s,\underline{X}_s) = E_{P_{s,x}}\left[f(t_o,\underline{X}_{t_o})\, C_s(\underline{X}) \,|\, \underline{\underline{A}}^s_s\right] \quad \text{a.s. } P_{s,x} \ .$$

Since under $P_{s,x}$ any set in $\underline{\underline{A}}^s_s$ has measure zero or one, the conditional expectation appearing above is the unconditional expectation and thus equals $v(s,x)$. Also $\underline{X}_s = x$ a.s. $P_{s,x}$. Hence we have

$$v'(s,\underline{X}_s) = v(s,x) \qquad \text{a.s. } P_{s,x} \ .$$

These observation imply

$$v'(s,x) = v(s,x) \ .$$

4. We now consider an equation dual to (2.24), namely

$$\frac{d}{dt} K_t = A^* K_t + g(t,.)K_t$$

where $\{K_t\} \subseteq \underline{\underline{M}}(S)$ - the class of finite signed measures on $(S,\underline{\underline{S}})$. The euqation (4.1) is purely formal and is to be interpreted as

(4.2) $$\langle f(t,.),K_t\rangle = \langle f(0,.),K_c\rangle + \int_0^t \langle Af(u,.),K_u\rangle du + \int_0^t \langle g(u,.)f(u,.),K_u\rangle du$$

for $f \in \underline{\underline{D}}_A$. Here, $\langle\theta,\mu\rangle$ denotes $\int \theta \, d\mu$ for $\mu \in \underline{\underline{M}}(S)$ and a function $\theta: S \to \mathbb{R}$. Thus $\langle f(t,.),\mu\rangle = \int f(t,x)d\mu(x)$ for $f \in \underline{\underline{J}}$. We will show that this equation with boundary condition

(4.3) $$K_o = \Pi \circ X_c^{-1}$$

admits a unique solution which is given by

(4.4) $$K_t(B) = E_\pi \left[1_B(X_t)\exp\left(\int_o^t g(u,X_u)du\right)\right], \quad B \in \underline{\underline{S}}.$$

The uniqueness will be proved in the class of $\{K_t\}$ satisfying

(4.5) $\{K_t\} \subseteq \underline{\underline{M}}(S)$, $t \to K_t(B)$ is a Borel measurable function for all $B \in \underline{\underline{S}}$ and $K_t \ll \Pi \circ X_t^{-1}$ with

$$\left| \frac{dK_t}{d\pi o X_t^{-1}} \right| \leq M$$

for all t, for a fixed constant M.

We continue to assume that the conditions imposed on (X_t) in Section 3, are valid. We further assume that S is a complete separable metric space. We begin with a Lemma.

<u>Lemma 5</u> : Let $0 < t < \infty$ be fixed. Let $\mu \in \underline{\underline{M}}(S)$ be such that

(4.6) $$\langle f(t,.),\mu\rangle = 0 \quad \forall f \in \underline{\underline{D}}_A .$$

Then $\mu \equiv 0$.

Proof : Let $\underline{\underline{E}}$ be the class of $f \in \underline{\underline{J}}$ for which (4.6) holds. Easy to see that if $f_k \in \underline{\underline{E}}$, $\text{w-}\lim_{k \to \infty} f_k = f$, then $f \in \underline{\underline{E}}$. Hence by (2.13), $\underline{\underline{J}}_o \subseteq \underline{\underline{E}}$.

For $f \in C_b(\hat{S})$, (i.e. $f : \hat{S} \to \mathbb{R}$ is bounded continuous), we have

$$(T_t\, f)(s,x) = E_{P_{s,x}}\, f(s+t, \underline{X}_{s+t}) \to f(s,x) \quad \text{as} \quad t \downarrow 0,$$

since $\underline{X}_u$ is right continuous. Thus $C_b(\hat{S}) \subseteq \underline{\underline{J}}_o \subseteq \underline{\underline{E}}$.

Given $f_o \in C_b(S)$, taking $f(s,x) = f_o(x)$, we have $f \in C_b(\hat{S}) \subseteq \underline{\underline{E}}$ and hence

$$(4.7) \qquad \langle f_o, \mu \rangle = 0 .$$

The validity of (4.7) for all $f_o \in C_b(S)$ implies $\mu = 0$ because $\underline{\underline{S}}$ - the Borel σ field - is also the smallest σ field with respect to which $C_b(S)$ is measurable.

$\square$

We are now in a position to prove the assertions made at the beginning of this section. This result may be considered as a dual Feynman-Kac's formula.

Theorem 6 : Suppose that g satisfies (2.17) and (2.18). Then the equation (4.2) with boundary condition (4.3) admits a unique solution in the class of $\{K_t\}$ satisfying (4.5). The unique solution is given by (4.4).

Proof : First we will prove that $\{K_t\}$ defined by (4.4) satisfies (4.2). Let $\{K_t\}$ be defined by (4.4). Easy to see that (4.3) and (4.5) are satisfied.

Taking $s = 0$ in Theorem 2, it follows from the martingale property of Z_t that $E_\pi Z_t = E_\pi Z_o$. Here, Z_t is given by (2.20) where in turn B_t is given by (2.19), with $s = 0$. Noting that with these notations,

$$< \theta, K_t > = E_\pi \theta(X_t)B_t$$

we conclude from the relation $E_\pi Z_t = E_\pi Z_o$ that

$$<f(t,.),K_t> - \int_0^t <(Af+\sigma f)(u,.),K_u>du = <f(0,.),K_o> .$$

Hence $\{K_t\}$ satisfies (4.2).

To prove the uniqueness part, we will prove the following. Suppose $\{K_t\}$ satisfies (4.2), (4.5) and $K_o \equiv 0$. Then $K_t \equiv 0$, $t \geq 0$.

For this fix $t_o < \infty$ and $f \in \underline{\underline{D}}_A$. Let ν be the measure defined on $S' = [0,t_o] \times S$ by

(4.8) $$\nu(B) = E_\pi \int_0^{t_o} 1_B(u, X_u)du, \qquad B \in \underline{\underline{S}}' = \underline{\underline{B}}(S').$$

Then note that (2.17) implies $\int_{S'} |g| \, d\nu < \infty$. Hence if $g_k : S' \to IR$ is defined by

(4.9) $$g_k(s,x) = g(s,x) \, 1_{\{|g(s,x)| \leq k\}}$$

then we have

(4.10) $$\int_{S'} |g_k - g| d\nu \to 0 \qquad \text{as} \qquad k \to \infty .$$

For each k, g_k is bounded by k. By Lusin's theorem (see [1], p. 187) we can get $c_{k,i} \in C_b(S')$, bounded by k, such that

(4.11) $$c_{k,i} \to g_k \qquad \text{a.e. } \nu \quad \text{as} \quad i \to \infty .$$

Hence

(4.12) $$\lim_{i \to \infty} \int |c_{k,i} - g_k| d\nu = 0.$$

Let $v_{k,i}$ be given by (3.6) for $c = c_{k,i}$. Then $A v_{k,i} = - c_{k,i} v_{k,i}$ on $[0, t_0) \times S$ by Theorem 4. Using (4.2) for $v_{k,i}$ and recalling that $K_0 = 0$, we have

(4.13) $$\begin{aligned} <f(t_0,.), K_{t_0}> &= < v_{k,i}(t_0,.), K_{t_0}> \\ &= \int_0^{t_0} <(Av_{k,i} + gv_{k,i})(u,.), K_u> du \\ &= \int_0^{t_0} <(g-c_{k,i})v_{k,i}(u,.), K_u> du . \end{aligned}$$

Thus

(4.14) $$\begin{aligned} |<f(t_0,.),K_{t_0}>| &\leq M \int_0^{t_0} <|(g-c_{k,i})v_{k,i}|(u,.), \pi \circ X_u^{-1} > du \\ &= M\, E_\pi \int_0^{t_0} |(g-c_{k,i})v_{k,i}|(u,X_u)du. \\ &= M \int |g-c_{k,i}|\cdot|v_{k,i}| d\nu . \end{aligned}$$

As $i \to \infty$, $v_{k,i}$ converges pointwise to v_k and is bounded by k, where v_k is given by (3.6) for $c = g_k$. This and (4.11), (4.12), (4.14) imply

$$|<f(t_0,.),K_{t_0}>| \leq M \int |g-g_k|\cdot|v_k| \, d\nu .$$

Since (4.9) implies $g_k(u,x) \leq a(u)$, it follows that

$$|v_k| \leq M_1 \cdot \exp(\int_0^{t_0} a(u)du) = M_2$$

where $|f| \leq M_1$. Hence

$$|\langle f(t_o,.), K_{to}\rangle| \leq M.M_2 \int |g-g_k| d\nu .$$

This and (4.10) imply $\langle f(t_o,.),K_{t_o}\rangle = 0$. Since $f \in \underline{\underline{D}}_A$ is arbitrary, Lemma 5 gives $K_t = 0$. This completes the proof.

□

We will briefly consider the equation for normalized measures

$$(4.15) \qquad N_t(B) = \frac{K_t(B)}{K_t(S)}, \qquad B \in \underline{\underline{S}}$$

where K_t is given by (4.4). It is easy to see, using (4.2) that $\{N_t\}$ satisfies.

$$(4.16) \quad \langle f(t,.),N_t\rangle = \langle f(0,.),N_o\rangle + \int_0^t \langle (Af+gf)(u,.),N_u\rangle du - \int_0^t \langle f(u,.),N_u\rangle\langle g(u,.),N_u\rangle du$$

We will now prove that $\{N_t\}$ is the unique solution to this equation.

<u>Theorem 7</u> : The equation (4.16) with boundary condition $N_o = \pi \circ X_o^{-1}$ admits a unique solution in the class of $\{N_t\}$ satisfying (4.5). The solution is given by (4.15).

<u>Proof</u> : We need to prove uniqueness of the solution. Let N_t' be any other solution, i.e. satisfying (4.5), (4.16) and $N_o' = \pi \circ X_o^{-1}$. Then it can be checked that $N_t'(S) = 1$ for all $t \geq 0$. Further, if K_t' is defined by

$$(4.17) \qquad K_t'(B) = N_t'(B) \cdot \exp\left(\int_o^t \langle g(u,.), N_u'\rangle du\right),$$

then K_t' is a solution to (4.2) and that it satisfies (4.3), (4.5). Hence by Theorem 6, $K_t' = K_t$. This and the observation that

$$N_t'(B) = \frac{K_t'(B)}{K_t'(S)}$$

give us the required equality, namely $N_t' = N_t$.

□

5. We will now give applications of the results in the previous sections to filtering theory.

We refer the reader to [4] for a detailed discussion and background on the white noise approach to filtering theory.

We assume that the signal process (X_t) is a Markov process satisfying the conditions imposed in the previous sections.

Let $\underline{\underline{K}}$ be a separable Hilbert space. Let $h : [0,T] \times S \to \underline{\underline{K}}$ be a measurable function such that

$$(5.1) \qquad E_\pi \left[\int_0^T \|h_u(X_u)\|^2_{\underline{\underline{K}}} \, du \right] < \infty$$

Let $H = L^2([0,T], \underline{\underline{K}})$ and let $\xi : \Omega \to H$ be defined by

$$(\xi(\omega))_u = h_u(X_u(\omega)), \quad 0 \leq u \leq T .$$

Consider the model

$$y = \xi + e$$

where $e = (e_t)$ is $\underline{\underline{K}}$-valued white noise independent of (X_t). Here y is the observation process and y, ξ, e are realised on a Quasi cylinder probability space $(E, \underline{\underline{E}}, \alpha)$ (See [4] section 6). We now state the Bayes formula. For the relevent definitions and proof, see [4].

<u>Theorem 8</u> : For $g : S \to IR$ bounded, measurable,

$$(5.2) \qquad E_\alpha(g(X_t) | y_u : u \leq t) = \int_S g(x) \, dF_t(y)(x)$$

where

$$(5.3) \qquad F_t(y)(B) = E_\pi \left[1_B(X_t) \exp\left(\int_0^t (h_u(X_u), y_u)_{\underline{\underline{K}}} du - \frac{1}{2} \int_0^t \|h_u(X_u)\|^2_{\underline{\underline{K}}} du \right) \right]$$

and

$$(5.4) \qquad F_t(y)(B) = \frac{\Gamma_t(y)(B)}{\Gamma_t(y)(S)}$$

for $0 \leq t \leq T$, $y \in H$, $B \in \underline{\underline{S}}$.

$\Gamma_t(y)$, $F_t(y)$ are known as unnormalized and normalized conditional distribution of X_t given $\{y_u : u \leq t\}$, respectively.

The following is an immediate consequence of Theorems 6,7. Let $g_y(t,x) = (h_t(x), y_t)_{\underline{\underline{K}}} - \frac{1}{2}||h_t(x)||^2_{\underline{\underline{K}}}$, $(t,x) \in [0,T] \times S$, $y \in H$.

__Theorem 9__ : Let h satisfy (5.1). (i) For all $y \in H$, $\Gamma_t(y)$ is the unique solution to the equation

$$(5.5) \qquad \langle f(t,.),\Gamma_t(y)\rangle = \langle f(0,.), \Gamma_0(y)\rangle + \int_0^t \langle (Af+g_y f)(u,.),\Gamma_u(y)\rangle du, \quad f \in \underline{\underline{D}}_A$$

with the condition $\Gamma_0(y) = \pi \circ X_0^{-1}$ in the class of $\{K_t\}$ satisfying (4.5).

(ii) For all $y \in H$, $F_t(y)$ is the unique solution to the equation

$$(5.6) \quad \langle f(t,.),F_t(y)\rangle = \langle f(0,.),F_0(y)\rangle + \int_0^t \langle (Af+g_y f)(u,.),F_u(y)\rangle du - \int_0^t \langle f(u,.),F_u(y)\rangle\langle g_y(u,.),F_u(y)\rangle du, \quad f \in \underline{\underline{D}}_A$$

with the initial condition $F_0(y) = \pi \circ X_0^{-1}$ in the class of $\{K_t\}$ satisfying (4.5).

__Proof__ : Since

$$|g_y(t,x)| \leq ||h_t(x)||^2_{\underline{\underline{K}}} + \frac{1}{2}||y_t||^2_{\underline{\underline{K}}}$$

and

$$g_y(t,x) \leq \frac{1}{2}||y_t||^2_{\underline{\underline{K}}} \quad ,$$

it follows that for all $y \in H$, g_y satisfies (2.17) and (2.18). Thus (i) follows from Theorem 6 and (ii) from Theorem 7.

Remark : Theorem 9 was proved in [3] under the much stronger condition

$$(5.7) \qquad ||h_t(x)|| \leq a_t \quad \text{with} \int_0^T a_t^2 \, dt < \infty .$$

The equations (5.5) and (5.6) are analogues of the Zakai and Fujisaki-Kallianpur-Kunita equations. In [3], $\Gamma_t(y)$ and $F_t(y)$ were also characterized as unique solutions to another type of equations (equations (3.4) and (3.11) in [3]) under the condition (5.7). With a little bit of work, it can be shown that (5.7) can be replaced by (5.1) in these results as well.

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